

## TWISTED SPACE-TIME SYMMETRY, NON-COMMUTATIVITY AND PARTICLE DYNAMICS

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We describe the twisted space-time symmetries which imply the quantum Poincaré covariance of noncommutative Minkowski spaces, with constant, Lie algebraic and quadratic commutators. Further we present the relativistic and nonrelativistic particle models invariant respectively under twisted relativistic and twisted Galilean symmetries.

### 1. Introduction

Since the work of Doplicher et all. (see e.g.<sup>1,2</sup>) there is a strong indication that due to quantum gravity effects the space-time coordinates are becoming noncommutative. In general case one can write<sup>a</sup>

$$\begin{aligned} [\hat{x}_\mu, \hat{x}_\nu] &= \frac{i}{\kappa^2} \theta_{\mu\nu}(\kappa \hat{x}_\rho) \\ &= \frac{i}{\kappa^2} \theta_{\mu\nu}^{(0)} + \frac{i}{\kappa} \theta_{\mu\nu}^{(1)\rho} \hat{x}_\rho + i \theta_{\mu\nu}^{(2)\rho\tau} \hat{x}_\rho \hat{x}_\tau, \end{aligned} \quad (1)$$

where the fundamental mass parameter  $\kappa$  has been introduced in order to exhibit the mass dimensions of respective terms and have the constant tensors  $\theta_{\mu\nu}^{(0)}$ ,  $\theta_{\mu\nu}^{(1)\rho}$ ,  $\theta_{\mu\nu}^{(2)\rho\tau}$  as dimensionless. If we link (1) with quantum gravity one can put  $\kappa = m_{\text{pl}}$  ( $m_{\text{pl}}$  - Planck mass). Further we add that the relation (1) describes in D=10 first-quantized open string theory the noncommutative coordinates on D-branes providing the localizations of the ends of the strings<sup>3,4</sup>.

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<sup>a</sup>Formula (1) is not the most general one. One can assume that the rhs of (1) depends also on momenta (or derivative operators) as well as on other operators, e.g. spin variables. In this note we shall not consider such extensions of (1). The expansion (1) is only up to quadratic term because higher orders do not have classical limit  $\kappa \rightarrow \infty$ .

There are two important problems related with the application of formula (1) to physical models:

- i) In standard relativistic theory, with classical Poincaré symmetries, the first term on rhs of (1) breaks the Lorentz invariance, and further two terms break both Lorentz and translational invariance. One can ask how looks the deformation of classical Poincaré invariance which permits to consider relations (1) as covariant under deformed Poincaré transformations, i.e. the same in any deformed Poincaré frame.
- ii) There should be given prescriptions how to formulate the classical mechanics and field theory models with noncommutative space-time coordinates (1), covariant under the twisted Poincaré symmetries.

If the time coordinate remains classical (i.e. in formula (1)  $\theta_{0\mu} = 0$ ) both points i) and ii) can be applied to the nonrelativistic noncommutative theories with classical Galilean invariance broken by relation (1).

## 2. Twisted Space-Time Symmetries

We shall look for the quantum relativistic symmetries implying the covariance of noncommutative Minkowski spaces. In systematic study firstly one should consider all possible quantum relativistic symmetries (quantum Poincaré algebras) in the form of noncommutative Hopf algebras, and then derive corresponding quantum Minkowski spaces as deformed Hopf algebra modules. An example of such a construction which is already more than ten years old is the  $\kappa$ -deformed Minkowski space<sup>5,6,7</sup>

$$[\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa} \hat{x}_i, \quad [\hat{x}_i, \hat{x}_j] = 0, \quad (2)$$

corresponding in (1) to the choice  $\theta_{\mu\nu}^{(0)} = \theta_{\mu\nu}^{(2)\rho\tau} = 0$  and  $\theta_{\mu\nu}^{(1)\rho} = \eta_{\mu 0} \delta_{\nu}^{\rho} - \eta_{\nu 0} \delta_{\mu}^{\rho}$ . Using the Hopf-algebraic formulae of  $\kappa$ -deformed Poincaré algebra in bicrossproduct basis one can show<sup>6</sup> that the relations (2) are covariant under the Hopf-algebraic action of  $\kappa$ -deformed Poincaré algebra.

It appears that the most effective way of describing the noncommutative space-times covariant under quantum relativistic symmetries is to consider twisted symmetry algebras. In such a case the classical Poincaré-Hopf algebra is modified only in the coalgebraic sector, with all the algebraic relations preserved. We change the classical Poincaré Hopf algebra  $\mathcal{H}^{(0)} = (\mathcal{U}(\mathcal{P}_4), m, \Delta_0, S_0, \epsilon)$  into twisted Poincaré Hopf algebra  $\mathcal{H} = (\mathcal{U}(\mathcal{P}_4), m, \Delta, S, \epsilon)$  by means of the twist factor  $\mathcal{F} \in \mathcal{U}(\mathcal{P}_4) \otimes \mathcal{U}(\mathcal{P}_4)$  as follows ( $\mathcal{P}_4 \ni \hat{g} = (P_\mu, M_{\mu\nu})$ )

$$\Delta(\hat{g}) = \mathcal{F} \circ \Delta_0(\hat{g}) \circ \mathcal{F}^{-1}, \quad S(\hat{g}) = U S_0(\hat{g}) U^{-1}, \quad (3)$$

$$\Delta_0(\hat{g}) = \hat{g} \otimes 1 + 1 \otimes \hat{g}, \quad S_0(\hat{g}) = -\hat{g}, \quad \epsilon(\hat{g}) = 0, \quad (4)$$

where  $(a \otimes b) \circ (c \otimes d) = ac \otimes bd$ . The twist  $\mathcal{F}$  satisfies the cocycle and normalization conditions<sup>8</sup>

$$\mathcal{F}_{12} (\Delta_0 \otimes 1) \mathcal{F} = \mathcal{F}_{23} (1 \otimes \Delta_0) \mathcal{F}, \quad (\epsilon \otimes 1) \mathcal{F} = (1 \otimes \epsilon) \mathcal{F} = 1, \quad (5)$$

where  $\mathcal{F}_{12} = f_{(1)} \otimes f_{(2)} \otimes 1$  etc. ( $\mathcal{F} = f_{(1)} \otimes f_{(2)}$ ) and  $U = f_{(1)} S(f_{(2)})$ .

The advantage of using twisted Poincaré algebra is the explicit formula for the multiplication in twisted Hopf algebra module  $\mathcal{A}$  which should satisfy the condition (see e.g.<sup>9</sup>,  $h \in \mathcal{U}(\mathcal{P}_4)$ ,  $a, b \in \mathcal{A}$ )

$$h \triangleright (a \bullet b) = (h_1 a) \bullet (h_2 b), \quad (6)$$

where  $\Delta(h) = h_1 \otimes h_2$ . We see from (6) that if  $h_1 \neq h_2$  then  $a \bullet b \neq b \bullet a$ , i.e. from quantum-deformed relativistic symmetry follow necessarily the noncommutative Minkowski space as its Hopf-algebraic module.

One can show that the multiplication in  $\mathcal{A}$  for twisted Hopf algebra  $\mathcal{H}$  which is consistent with the relation (6) ( $h \in \mathcal{H}$ ) provides the formula<sup>10,11,12</sup>

$$a \bullet b = (\bar{f}_{(1)} a) (\bar{f}_{(2)} b), \quad \mathcal{F}^{-1} = \bar{f}_{(1)} \otimes \bar{f}_{(2)}. \quad (7)$$

In the case of relativistic symmetries one can use the classical space-time representation for the Poincaré generators  $P_\mu$ ,  $M_{\mu\nu}$

$$P_\mu = i\partial_\mu, \quad M_{\mu\nu} = i(x_\nu\partial_\mu - x_\mu\partial_\nu). \quad (8)$$

Subsequently in the formula (7) one can assume that  $a$ ,  $b$  are classical functions on commutative Minkowski space  $x_\mu$ , and define  $\bar{f}_{(i)}(P_\mu, M_{\mu\nu}) \equiv \bar{f}_{(i)}(x, \partial)$ ,  $i = 1, 2$ . One gets the following star product multiplication which is a particular representation of algebraic formula (7)

$$\xi(x) \star \zeta(x) = (\bar{f}_{(1)}(x, \partial)\xi(x)) (\bar{f}_{(2)}(x, \partial)\zeta(x)). \quad (9)$$

The important application of twisted Poincaré algebras to the covariant description of noncommutative Minkowski spaces, namely describing the quantum covariance of (1) for the case  $\theta_{\mu\nu} = \theta_{\mu\nu}^{(0)}$  is quite recent<sup>b</sup>. The quantum symmetry which leaves invariant the simplest form of (1)<sup>c</sup>

$$[\hat{x}_\mu, \hat{x}_\nu]_\bullet = \frac{i}{\kappa^2} \theta_{\mu\nu}^{(0)}, \quad (10)$$

<sup>b</sup>The twisted Poincaré symmetries corresponding to  $\theta_{\mu\nu} = \theta_{\mu\nu}^{(0)}$  were earlier discussed in 13,14,15,16, but the full consequences of the twisted description were realized in 2004 (see e.g. 12,17,18,19).

<sup>c</sup>Below, in chapter 2 and 3, we shall use explicitly the *fat dot notation* for the algebra of functions on quantum Minkowski space in order to stress its Hopf algebra module origin.

(where  $[a, b]_\bullet = a \bullet b - b \bullet a$ ) is generated by the following Abelian twist

$$\mathcal{F}_\theta = \exp \frac{i}{2\kappa^2} (\theta_{(0)}^{\mu\nu} P_\mu \wedge P_\nu). \quad (11)$$

We obtain the twisted Poincaré-Hopf structure with classical Poincaré algebra relations and modified coproducts of Lorentz generators  $M_{\mu\nu}$

$$\Delta_\theta(P_\mu) = \Delta_0(P_\mu), \quad (12)$$

$$\begin{aligned} \Delta_\theta(M_{\mu\nu}) &= \mathcal{F}_\theta \circ \Delta_0(M_{\mu\nu}) \circ \mathcal{F}_\theta^{-1} \\ &= \Delta_0(M_{\mu\nu}) - \frac{1}{\kappa^2} \theta_{(0)}^{\rho\sigma} [(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \otimes P_\sigma \\ &\quad + P_\rho \otimes (\eta_{\sigma\mu} P_\nu - \eta_{\sigma\nu} P_\mu)]. \end{aligned} \quad (13)$$

One can consider however also other Abelian twists of Poincaré symmetries, depending on the Lorentz generators  $M_{\mu\nu}$  (see <sup>20,13,14,21</sup>). It appears that only subclass of general commutator (1) with linear and quadratic terms can be covariantized by twisted Poincaré algebras. In the following section we shall consider the quantum Poincaré symmetries corresponding to the following two twist functions <sup>21</sup>:

i) Lie-algebraic relations for noncommutative Minkowski space

$$\mathcal{F}_{(\alpha\beta)} = \exp \frac{i}{2\kappa} (\zeta^\lambda P_\lambda \wedge M_{\alpha\beta}), \quad (14)$$

where  $\alpha, \beta = 0, 1, 2, 3$  are fixed and the vector  $\zeta^\lambda = \theta_{(1)}^{\lambda\alpha\beta}$  has vanishing components  $\zeta^\alpha, \zeta^\beta$ .

ii) Quadratic deformations of Minkowski space

$$\mathcal{F}_{(\alpha\beta\gamma\delta)} = \exp \frac{i}{2} \zeta M_{\alpha\beta} \wedge M_{\gamma\delta}, \quad (15)$$

where  $\zeta = \theta_{(2)}^{\alpha\beta\delta\gamma}$  is a numerical parameter, all the four indices  $\alpha, \beta, \gamma, \delta$  are fixed and different.

### 3. Lie-algebraic and Quadratic Quantum-Covariant Noncommutative Minkowski Spaces

In this Section we shall report on results presented in <sup>21</sup>, which we supplement by the proof of quantum translational invariance.

In the formalism of quantum-deformed Hopf-algebraic symmetries the quantum-covariant noncommutative Minkowski space can be introduced in two ways:

i) as the translation sector of quantum Poincaré group ,

ii) as the quantum representation space (a Hopf algebra module) for quantum Poincaré algebra with the action of the deformed symmetry generators satisfying suitably deformed Leibnitz rule (6).

In the case of constant tensor  $\theta_{\mu\nu} = \theta_{\mu\nu}^{(0)}$  the quantum Poincaré group algebra dual to the coproducts (12), (13) is known<sup>15,19,21</sup>, and the quantum translations do not satisfy the relation (10). It appears that the relation (10) as describing quantum-covariant noncommutative Minkowski space can be obtained only as the Hopf algebra module. To the contrary, in the case of twisted relativistic symmetries generated by the twist factors (14, 15) it can be shown that both definitions i) and ii) coincide<sup>21</sup>.

i) Lie-algebraic noncommutative Minkowski space.

The commutator algebra following from (14) and the formula (7) has the form<sup>21</sup>

$$[\hat{x}_\mu, \hat{x}_\nu]_\bullet = C_{\mu\nu}^\rho \hat{x}_\rho, \quad (16)$$

where

$$C_{\mu\nu}^\rho = \frac{i}{\kappa} \zeta_\mu (\eta_{\beta\nu} \delta_\alpha^\rho - \eta_{\alpha\nu} \delta_\beta^\rho) + \frac{i}{\kappa} \zeta_\nu (\eta_{\alpha\mu} \delta_\beta^\rho - \eta_{\beta\mu} \delta_\alpha^\rho). \quad (17)$$

The relations (16) can be written in more transparent way as follows ( $\alpha, \beta$  are fixed by the choice of twist function)

$$[\hat{x}_\alpha, \hat{x}_\lambda]_\bullet = \frac{i}{\kappa} \zeta_\lambda \eta_{\alpha\alpha} \hat{x}_\beta, \quad [\hat{x}_\beta, \hat{x}_\lambda]_\bullet = -\frac{i}{\kappa} \zeta_\lambda \eta_{\beta\beta} \hat{x}_\alpha, \quad (18)$$

where  $\zeta_\alpha = \zeta_\beta = 0$ .

The quantum Lorentz covariance of (16) under the Hopf action of the Lorentz generators  $M_{\mu\nu}$  has been shown in<sup>21</sup>. We shall show the quantum translational invariance of (16) using the differential realization (8). The fourmomentum coproduct generated by twist (14) has the form<sup>21</sup>

$$\Delta(P_\mu) = \Delta_0(P_\mu) + \frac{1}{2\kappa} \xi^\lambda P_\lambda \wedge (\eta_{\alpha\mu} P_\beta - \eta_{\beta\mu} P_\alpha) + \mathcal{O}(P^3). \quad (19)$$

Putting in (6)  $h \equiv P_\mu$ ,  $a \equiv x_\rho$ ,  $b \equiv x_\sigma$  and using (17) we obtain

$$\begin{aligned} P_\mu \triangleright (x_\rho \bullet x_\sigma) &= ix_{\{\rho} \eta_{\sigma\}\mu} + \eta_{\alpha\mu} \xi_{[\sigma} \eta_{\rho]\beta} - \eta_{\beta\mu} \xi_{[\sigma} \eta_{\rho]\alpha}, \\ &= ix_{\{\rho} \eta_{\sigma\}\mu} + \frac{1}{2} P_\mu \triangleright C_{\rho\sigma}^\lambda x_\lambda. \end{aligned} \quad (20)$$

Finally we get

$$P_\mu \triangleright [x_\rho, x_\sigma]_\bullet = P_\mu \triangleright C_{\rho\sigma}^\lambda x_\lambda, \quad (21)$$

i.e. the relation (16) is covariant.

ii) Quadratic noncommutativity of Minkowski space coordinates.

After using the formula (7) with inserted twist (15) one gets the following commutation relations of space-time coordinates ( $\{a, b\}_\bullet = a \bullet b + b \bullet a$ )

$$\begin{aligned} [\hat{x}_\mu, \hat{x}_\nu]_\bullet &= i \sinh \frac{\zeta}{2} \cosh \frac{\zeta}{2} (\eta_{\alpha[\mu} \eta_{\gamma\nu]} \{\hat{x}_\beta, \hat{x}_\delta\}_\bullet - \eta_{\alpha[\mu} \eta_{\delta\nu]} \{\hat{x}_\beta, \hat{x}_\gamma\}_\bullet \\ &\quad - \eta_{\beta[\mu} \eta_{\gamma\nu]} \{\hat{x}_\alpha, \hat{x}_\delta\}_\bullet + \eta_{\beta[\mu} \eta_{\delta\nu]} \{\hat{x}_\alpha, \hat{x}_\gamma\}_\bullet) \\ &\quad - \sinh^2 \frac{\zeta}{2} \left( \sum_{\substack{k=\alpha, \beta \\ l=\gamma, \delta}} \delta^k_{[\mu} \delta^l_{\nu]} [\hat{x}_k, \hat{x}_l]_\bullet \right), \end{aligned} \quad (22)$$

or in more explicit form ( $k = \alpha, \beta$  and  $l = \gamma, \delta$ )

$$\begin{aligned} [\hat{x}_k, \hat{x}_l]_\bullet &= i \tanh \frac{\zeta}{2} (\eta_{\alpha k} \eta_{\gamma l} \{\hat{x}_\beta, \hat{x}_\delta\}_\bullet - \eta_{\alpha k} \eta_{\delta l} \{\hat{x}_\beta, \hat{x}_\gamma\}_\bullet \\ &\quad - \eta_{\beta k} \eta_{\gamma l} \{\hat{x}_\alpha, \hat{x}_\delta\}_\bullet + \eta_{\beta k} \eta_{\delta l} \{\hat{x}_\alpha, \hat{x}_\gamma\}_\bullet), \end{aligned} \quad (23)$$

and  $[\hat{x}_\alpha, \hat{x}_\beta]_\bullet = [\hat{x}_\gamma, \hat{x}_\delta]_\bullet = 0$ .

We conjecture that the relations (22) are covariant under the action of quantum Poincaré symmetries, generated by twist (15).

The linear and quadratic relations (18) and (23) provide special choices of the constant parameters  $\theta_{\mu\nu}^{(1)\rho}$ ,  $\theta_{\mu\nu}^{(2)\rho\tau}$  for which the quantum covariance group was found in <sup>21</sup>.

#### 4. Particle Dynamics Invariant Under Twisted Relativistic and Galilean Symmetries

The discussion of the noncommutative dynamical theories one begins naturally with the consideration of classical mechanics models. We shall restrict our considerations here to the case  $\theta_{\mu\nu} = \theta_{\mu\nu}^{(0)}$ , i.e. the noncommutative space-time described by (10). One can introduce the Lagrangian models describing free point particles moving in noncommutative space-time in the following two ways:

i) If  $\theta_{\mu 0} = 0$ , i.e. we have the relations

$$[\hat{x}_i, \hat{x}_j] = i \theta_{ij}, \quad (24)$$

$$[\hat{x}_0, \hat{x}_i] = 0, \quad (25)$$

we deal with classical time variable  $t$ , where  $\hat{x}_0 = ct$  and noncommutative space coordinates  $\hat{x}_i$ . In such a case one can look for the non-relativistic Lagrangian models with constraints, which provide the relation (24) as the quantized Dirac bracket. Such a first model

was constructed in <sup>22</sup> in  $D = (2 + 1)$  dimensions with the following Lagrangian

$$\mathcal{L} = \frac{m\dot{x}_i^2}{2} - k\epsilon_{ij}\dot{x}_i\ddot{x}_j. \quad (26)$$

The higher order Lagrangian (26) can be expressed if first order form in six-dimensional phase space  $(x_i, p_i, \tilde{p}_i)$ <sup>d</sup> and after introducing the linear transformations

$$\begin{aligned} X_i &= x_i - \frac{2}{m}\tilde{p}_i, \\ P_i &= p_i, \\ \tilde{P}_i &= \epsilon_{ij}\tilde{p}_j + \frac{k}{m}p_i, \end{aligned} \quad (27)$$

one obtains the following symplectic structure for the variables  $Y_A = (X_i, P_i, \tilde{P}_i)$ ,  $(A = 1\dots 6)$

$$\{Y_A, Y_B\} = \Omega_{AB}, \quad \Omega = \begin{pmatrix} \frac{2k}{m^2}\epsilon & 1_2 & 0 \\ -1_2 & 0 & 0 \\ 0 & 0 & \frac{k}{2}\epsilon \end{pmatrix}. \quad (28)$$

One can identify (24) with quantized PB for the space variables  $X_i$  if we put in (24)  $\theta_{ij} = \frac{2k}{m^2}\epsilon_{ij}$ .

In <sup>22</sup> the dimension  $D = 2 + 1$  was chosen because in two space dimensions one can put  $\theta_{ij} = \theta\epsilon_{ij}$ , i.e. the relation (24) does not break the classical Galilean invariance. However if  $k \neq 0$  the Galilean algebra is centrally extended by second *exotic* central charge <sup>23</sup>.

- ii) For general constant  $\theta_{\mu\nu}$  one obtains the noncommutative action describing free particle motion if we introduce in the first order action for classical massive relativistic particle

$$S = \int d\tau [y_\mu p^\mu - e(p^2 - m^2)], \quad (29)$$

the following change of variables (we recall that  $\theta_{\mu\nu} = \theta_{\mu\nu}^{(0)}$ )

$$y_\mu = x_\mu + \frac{1}{a}\theta_{\mu\nu}p^\nu. \quad (30)$$

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<sup>d</sup>The momenta  $p_i, \tilde{p}_i$  are described by the following formulae

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \ddot{x}_i}, \quad \tilde{p}_i = \frac{\partial \mathcal{L}}{\partial \ddot{x}_i}.$$

It is easy to check that if we introduce CCR following from (29)

$$[y_\mu, y_\nu] = 0, \quad [y_\mu, p^\nu] = i\delta_\mu^\nu, \quad [p^\mu, p^\nu] = 0, \quad (31)$$

then the variables  $x_\mu$  in (30) satisfy the relation (10) if we put  $a = 2\kappa^2$ . Using the relation (30) one can rewrite the action (29) as follows

$$S = \int d\tau [\dot{x}_\mu p^\mu - e(p^2 - m^2) + \frac{1}{a} \theta^{\mu\nu} \dot{p}_\mu p_\nu]. \quad (32)$$

The variables  $y_\mu, p_\mu$  in (29) are classical, i.e. transform under Lorentz rotations in standard way

$$y'_\mu = \Lambda_\mu^\nu y_\nu, \quad p'_\mu = p_\mu. \quad (33)$$

Using (30) and (33) one gets however

$$\begin{aligned} x'_\mu &= y'_\mu + \frac{1}{a} \theta_{\mu\nu} \Lambda_\nu^\rho p^\rho \\ &= \Lambda_\mu^\nu x'_\nu + \frac{1}{a} (\Lambda_\mu^\rho \theta_{\rho\nu} + \theta_{\mu\rho} \Lambda_\nu^\rho) p^\nu. \end{aligned} \quad (34)$$

Interestingly enough, the transformations (34) describe exactly the twisted Lorentz transformations, generated by the coproduct (13), which leave invariant the action (32) for the noncommutative relativistic particle.

The model (32) has been firstly obtained without reference to twisted Lorentz symmetries by Deriglazov <sup>24</sup> and its non-relativistic version

$$S_{NR} = \int dt [\dot{x}_i \dot{p}_i - \frac{1}{2m} \vec{p}^2 + \frac{1}{a} \theta_{ij} \dot{p}_i p_j], \quad (35)$$

in  $D = 2+1$ , when  $\theta_{ij} = \epsilon_{ij}$ , it was proposed by Duval and Horvathy <sup>25</sup>. It is well-known however that the model (35) can be also derived from the model (26). Indeed, the first order formulation of the model (26) in Faddeev-Jackiw approach <sup>26</sup> to higher order Lagrangians provides the action <sup>27,28</sup>

$$\mathcal{L} = p_i (\dot{x}_i - y_i) + \frac{\vec{y}^2}{2m} + \frac{1}{a} \epsilon_{ij} \dot{p}_i p_j. \quad (36)$$

The Lagrangian (36) after introducing the new coordinates

$$X_i = x_i + \frac{1}{a} \epsilon_{ij} (y_j - p_j), \quad (37)$$

provides the Lagrangian (35) (with  $x_i$  replaced by  $X_i$ ) and additional term which depends on auxiliary internal variables commuting with  $(X_i, p_i)$  <sup>27</sup>.

The nonrelativistic model (35) can be considered in any space dimension  $d$ . If  $d = 2$  the action (35) is, similarly as (26), invariant under the transformations of exotic  $(2+1)$ -dimensional Galilean group. If  $d > 2$  the invariance of the nonrelativistic model (35) can be achieved by considering quantum Galilean symmetries, with twisted space rotations generated by the following nonrelativistic twist

$$\mathcal{F}_\theta^{NR} = \exp \frac{i}{a} \theta_{ij} P_i \wedge P_j. \quad (38)$$

The formulation of twisted quantum mechanics invariant under twisted quantum Galilei group is now under our consideration.

### 5. Final Remarks

We presented in this paper some selected aspects of the theory of noncommutative space-times, with new results on quantum Poincaré covariance of a class of linearly and quadratically deformed Minkowski spaces. We also considered the non-relativistic and relativistic particle models on noncommutative space-time with numerical value of the noncommutativity function  $\theta_{\mu\nu} = \theta_{\mu\nu}^{(0)}$  and have pointed out their twisted quantum covariance. We see that the role of quantum deformations is to introduce in place of broken classical symmetries a modified transformations which imply the quantum covariance. Such a possibility selects only particular class of tensors  $\theta_{\mu\nu}^{(1)\rho}$  and  $\theta_{\mu\nu}^{(2)\rho\tau}$  in formula (1).

Most of the applications of the noncommutative space-times in the literature assume the choice  $\theta_{\mu\nu} = \theta_{\mu\nu}^{(0)}$  (see (10)). In this talk we presented also the results for linear ( $\theta_{\mu\nu}^{(1)\rho} \neq 0$ ) and quadratic ( $\theta_{\mu\nu}^{(2)\rho\tau} \neq 0$ ) deformations of Minkowski space. The extension of particle models on noncommutative space-times to linearly and quadratically deformed Minkowski spaces is now studied.

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